

Splitting of roton minimum in the $\nu = 5/2$ Moore-Read state

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We calculate the dynamical structure factor of the $\nu = 5/2$ non-abelian quantum Hall state in the dipole approximation, valid for large momenta. Due to the fact that both quasi-particles (qps) and quasi-holes (qhs) have an internal Majorana degree of freedom, a qp-qh pair has a fermionic degree of freedom which can be either empty or occupied, and leads to a splitting of the roton mode. Observation of this splitting by means of finite wavelength optical spectroscopy could provide evidence for Majorana modes in the $\nu = 5/2$ quantum Hall state.

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The $\nu = 5/2$ quantum Hall (QH) state is expected to support non-abelian quasi-particles (qps), which could be used for topological quantum computation [1]. There is both theoretical and experimental evidence that the $\nu = 5/2$ QH state is indeed described by the non-abelian Moore-Read state [2]. Most significantly, the quasiparticles in the Moore-Read state have charge $e/4$. This has been confirmed by tunneling [3], shot noise [4] and interference [5] experiments. However, photoluminescence experiments seem to indicate that the state is not spin polarized, contrary to the Moore-Read state [6]. Recent resonant light scattering experiments suggest that both polarized and unpolarized domains may form in the second Landau level in general [7]. Clearly more evidence is needed to confirm the validity of the Moore-Read state.

The roton mode is the collective excitation of quantum Hall states [8, 9]. In integer quantum Hall systems, it can be understood as a particle hole pair, in fractional quantum Hall (FQH) systems as a quasi-particle (qp) quasi-hole (qh) pair. For hierarchical FQH states, there is one roton minimum for each level of the hierarchy. The roton mode can be probed with the help of optical excitations, but only in the presence of finite wavelength density modulations. Using surface acoustic waves to create a charge density modulation, several roton minima for hierarchical FQH states were recently observed experimentally [10].

The non-abelian statistics of $\nu = 5/2$ qps is encoded by an internal Majorana degree of freedom. Here we show that as a direct consequence of these Majorana fermions, the roton mode in the $\nu = 5/2$ Moore-Read state splits into two branches. The two Majorana modes of a qp-qh pair can be combined into a neutral fermion, which can be either occupied or unoccupied. Due to the finite overlap of the two Majorana wave functions associated with qp and qh, an occupied neutral fermion has higher energy than an unoccupied one, hence the splitting of the roton mode. The splitting both oscillates and decays exponentially with increasing separation of the pair. The main result of this work is the excitation spectrum given by Eq. (10), whose form is shown in Fig. 1. We argue that the splitting of the roton mode and hence the non-abelian

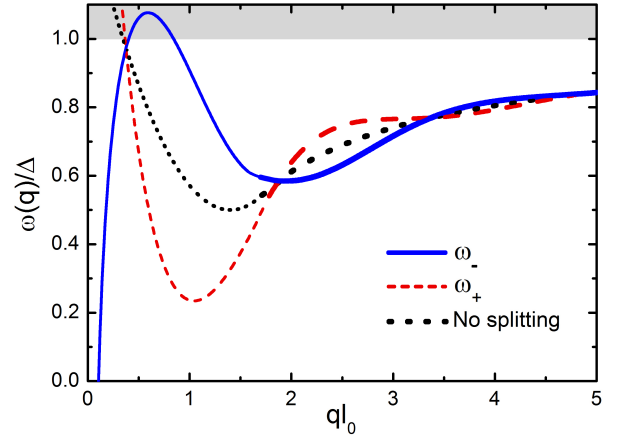


FIG. 1: (Color online) The expected roton dispersion. The thick lines are given by Eq. (10), constituting the dipole approximation, valid for q beyond the roton minimum, which occurs near $q l_0 \approx 1.4$ [14]. The usual single roton mode (shown with black dots) has split into two branches with distinct extrema, denoted ω_+ and ω_- , where $\omega_+ - \omega_- = 2E(2l_0^2 q)$. ω_- is the brighter primary branch, and ω_+ is the secondary split-off branch. In the small q -regime, the dipole Hamiltonian is no longer valid and the roton minimum is taken to be parabolic with appropriate parameters used as outlined in the text. This regime is plotted with thin lines. $\Delta \approx 0.025e^2/\epsilon l_0$ is the $\nu = 5/2$ exciton gap [15]. The grey shaded region denotes the two-roton continuum.

nature of the $\nu = 5/2$ state can be directly probed in an optical experiment similar to [10].

We model the Moore-Read state as a p-wave superconductor of composite fermions [11], and then use a Ginzburg-Landau approach to describe the dynamics of the superconducting order parameter. Charge $e/4$ qps of the $5/2$ state are vortices of the superconductor, and are treated in the same way as the vortices of condensed composite bosons [12, 13] in order to calculate the roton branch. The dynamics of the order parameter effective theory with N vortices are identical to those of an N point particle Hamiltonian with pairwise interactions

and a constant kinetic term per particle, with the commutation relations $[\hat{x}_i, \hat{y}_j] = -il_0^2 \delta_{i,j} / \nu$, where i denotes the i^{th} vortex, and $l_0 = \sqrt{\hbar/eB}$ is the magnetic length.

Vortex configurations of the order parameter in the Bogolubov-de-Gennes (BdG) Hamiltonian give rise to low-energy Majorana excitations, pairs of which form neutral fermions. Our Hilbert space then includes both the vortex degrees of freedom as well as the occupancy of neutral fermions that reside in pairs of vortices. We shall truncate the many-body Hilbert space of vortices to a single vortex-antivortex pair, precisely the qp-qh pair that forms the roton mode.

For Majoranas separated by r , the unoccupied and occupied energies of the neutral fermion have been calculated for a p-wave superconductor [16]

$$E(r) = \pm \frac{2\Delta_0}{\pi^{3/2}} \frac{\cos(p_F r + \pi/4)}{\sqrt{p_F r}} \exp\left(-\frac{r}{\xi}\right). \quad (1)$$

in which Δ_0 is the mean field value of the order parameter field, p_F is the Fermi velocity, and ξ is the Majorana coherence length. To make quantitative predictions for the Moore-Read state, we use the estimates of Ref. [17] for the magnitude of the splitting near the roton minimum ($\approx 0.01e^2/\epsilon l_0$), the coherence length, $\xi \approx 2.3l_0$, and for $p_F \approx \pi/3l_0$.

The Hamiltonian for a single qp-qh pair in this effective picture is

$$\hat{H}_0(\hat{r}) = \Delta + V(\hat{r}) + E(\hat{r})(2\hat{c}^\dagger \hat{c} - 1) \quad (2)$$

Where Δ is the energy gap to create two isolated qps [15], $\hat{r} = |\hat{\mathbf{r}}_1 - \hat{\mathbf{r}}_2|$ is the dipole separation operator, V is the Coulomb interaction between the charge $\pm e/4$ vortices, and $\hat{c}^\dagger = \hat{\gamma}_1 - i\hat{\gamma}_2$ is the neutral fermion creation operator which contains two Majorana operators ($\hat{\gamma}_1$ and $\hat{\gamma}_2$). The non-vanishing commutators in relative separation ($\hat{\mathbf{r}}$) and centre of mass ($\hat{\mathbf{R}}$) coordinates are $[\hat{r}_x, \hat{R}_y] = 2il_0^2$, and $[\hat{r}_y, \hat{R}_x] = -2il_0^2$. The canonical centre of mass momentum is then related to the relative separation as $\hat{\mathbf{r}} = 2l_0^2 \hat{\mathbf{z}} \times \hat{\mathbf{K}}$, where $\hat{\mathbf{z}}$ is the out-of-plane unit vector. The Hamiltonian has the dynamics of a dipole drifting with constant velocity in a magnetic field, with an internal 2-level degree of freedom.

For qp-qh separations comparable to the intrinsic size of a vortex, the dipole approximation breaks down, and the leading contribution to the dynamic structure factor is of quadrupole type. For separations much larger than the intrinsic size of a vortex, the single dipole approximation is asymptotically exact [13].

Formally, the ground state of the Hamiltonian Eq. (2) is at $r = 0$, with both a divergent splitting energy Eq. (1) and a divergent potential V . These divergences are an artefact of the point-particle description of vortices: the Hamiltonian Eq. (2) is only valid for well separated qps, ie. r greater than twice the intrinsic size of a single vortex. However, in the following calculation we use a completely

overlapping qp-qh pair as the vacuum state and assign the energy zero to it. Then, the density operator defined below separates the qp and qh from each other and creates a finite energy excitation described by Eq. (2).

The parity of the number of quasiparticles must be conserved: an excitation breaks a Cooper pair, necessarily incorporating *two* fermions. It is necessary then, to add to the Hamiltonian Eq. (2) a set of vortices pinned by disorder, which can, in pairs, support neutral fermions. These vortices will inevitably be present in any real system due to disorder, but their density can also be tuned externally.

The arrangement of pinned vortices is assumed to be random. We are interested in the dynamics of a single qp-qh pair, created by a momentum fluctuation within this background of pinned vortices. If the average separation of two pinned vortices is greater than a few times the qp-qh separation, the overlap of the dipole vortices with the disorder vortices will be appreciable for *at most* one pair of disorder sites. Thus, in the spirit of the random singlet phase [18], we can discard all other pinned vortices, and reduce our system to that of four vortices: the qp-qh pair and the two closest pinned vortices. Let the operators \hat{c} and \hat{c}^\dagger correspond to the dipole neutral fermion, and \hat{b} and \hat{b}^\dagger to the single pinned vortex pair. We can now introduce a tunneling term to the Hamiltonian,

$$\hat{H}_T = t_d \hat{b} \hat{c} - t_d^* \hat{c}^\dagger \hat{b}^\dagger + t'_d \hat{b}^\dagger \hat{c} - t_d'^* \hat{c}^\dagger \hat{b} + (2\hat{b}^\dagger \hat{b} - 1)\epsilon_d \quad (3)$$

Here, t_d and t'_d are the complex overlap integrals between the four sites (1, 2, 3, 4), and ϵ_d denotes the energy splitting of the localized neutral fermion. We obtain $t_d = E(|\mathbf{r}_1 - \mathbf{r}_3|) - E(|\mathbf{r}_2 - \mathbf{r}_4|) + i(E(|\mathbf{r}_2 - \mathbf{r}_3|) + E(|\mathbf{r}_1 - \mathbf{r}_4|))$, where t'_d has a similar form but is irrelevant here. We can now proceed to calculate the collective excitation spectrum, given by $\omega(q) = \int \omega S(q, \omega) d\omega / \int S(q, \omega) d\omega$ [9], where the dynamic structure factor is given by

$$S(q, \omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \hat{\rho}_q(t) \hat{\rho}_{-q}(0) \rangle. \quad (4)$$

Here $\hat{\rho}_q(t)$ is the charge density operator with time dependence $\hat{\rho}_q(t) = e^{i\hat{H}t} \hat{\rho}_q(0) e^{-i\hat{H}t}$, in which $\hat{H} = \hat{H}_0 + \hat{H}_T$. For a single qp-qh pair, $\hat{\rho}_q(0) = \sum_{s=\pm 1} s e^{i\mathbf{q} \cdot (\frac{s}{2}\hat{\mathbf{r}} + \hat{\mathbf{R}})}$. We calculate the eigenstates of H_0 explicitly, and perturbatively introduce H_T up to second order. Fourier transforming, we obtain two terms, $S_0(q, \omega)$ which contains only the unperturbed Hamiltonian, and $S_1(q, \omega)$ which contains \hat{H}_T to second order. The first term produces the primary excitation branch in which the neutral fermions are unoccupied. That the neutral fermions are unoccupied is immediately obvious as the charge density operator acts only on the vortex sector of the ground state Hamiltonian. The second term contains only those terms $\propto t_d t_d^*$, as the mixed creation-annihilation operator terms

vanish. This term in the structure factor contains operators that fill the two neutral fermion states, and produces the secondary split-off branch, peculiar to the $\nu = 5/2$ Moore-Read state.

The total Hilbert space is a product of the vortex space with matrix elements of the form $\langle 0|\hat{\rho}_q|\mathbf{K}\rangle$, and the neutral fermion space with matrix elements of the form $\int_0^t dt_1 \int_0^{t_1} dt_2 \langle 0|\hat{H}_T(t_1)\hat{H}_T(t_2)|m,n\rangle$, where m and n correspond to the occupancy of the disorder-pair and the dipole respectively, and the time dependence of $\hat{H}_T(t)$ arises from using the interaction picture with respect to \hat{H}_0 . The unusual temporal limits reflect that for times $t_{1(2)} < 0$ or $t_{1(2)} > t$, the system is in the vacuum state and so cannot support Majoranas in the dipole, making \hat{H}_T nonzero only in the time interval $[0, t]$. The vortex part of the ground state Hamiltonian is diagonal in relative separation ($\hat{\mathbf{r}}$), and the centre of mass coordinates do not appear. The wavefunction can be written as

$$\psi_{\mathbf{K}}(\mathbf{r}, \mathbf{R}) = \frac{1}{\sqrt{V}} e^{i\mathbf{K}\cdot\mathbf{R}} \delta(\mathbf{r} + 2l_0^2 \hat{z} \times \mathbf{K}). \quad (5)$$

This is a point particle description of the vortices. However, in order to calculate the overlap of two vortices, the vortices must be given a spatial profile whose width reflects the size of the vortex. We introduce a Gaussian profile, such that $\delta(\mathbf{r} + 2l_0^2 \hat{z} \times \mathbf{K}) \rightarrow 2l_0/\sqrt{\pi} e^{-|\mathbf{r} + 2l_0^2 \hat{z} \times \mathbf{K}|^2/4l_0^2}$. For a charge-density fluctuation of momentum $q\hat{x}$, we obtain a single solution which is a qp-qh pair with separation $2l_0^2 q\hat{y}$, and matrix element $2\exp(-l_0^2 q^2/2)$, thus obtaining

$$S_0(q, \omega) = \int_{-\infty}^{\infty} dt \int \frac{d\mathbf{K}}{4\pi^2} \sum_{m,n} e^{-i(\epsilon_{mnK} - \epsilon_{000} - \omega)t} \cdot |\langle 0, 0, 0|\hat{\rho}_q|m, n, \mathbf{K}\rangle|^2 \quad (6)$$

where the eigenvalues are

$$\epsilon_{mnK} = \Delta + V(2l_0^2 K) + (2n-1)E(2l_0^2 K) + (2m-1)\epsilon_d \quad (7)$$

and ϵ_d is the contribution from the pinned vortex Majoranas. As the density operator is simply the real space translation operator, the summation in the Majorana sector gives $\delta_{m,0}\delta_{n,0}$. Therefore this part of the structure factor is the primary branch of Eq. (9), without the decreased spectral weighting of M_Q^2 . As for $S_1(q, t)$, the first order expansion of H_T gives zero, and for the second order term we find

$$S_1(q, \omega) = \frac{t_d t_d^*}{4} \int dt dt_1 dt_2 \sum_{m,n} \int \frac{d^2 K}{4\pi^2} e^{-i(\epsilon_{mnK} - \epsilon_{00K})(t_1 - t_2)} e^{-i(\epsilon_{00K} - \epsilon_{000} + \omega)t} |\langle 0, 0, 0|\hat{\rho}_q \hat{c}\hat{b}|m, n, \mathbf{K}\rangle|^2 \quad (8)$$

Here the summation over the Majorana sector results in a single occupied neutral fermion term $\delta_{m,1}\delta_{n,1}$ due to the

operator pair $\hat{c}\hat{b}$ and its complex conjugate. Therefore this term contributes the delta functions in Eq. (9). We thus obtain the structure factor, and therefore excitation spectrum, of our qp-qh pair system where the vortices support neutral fermions with non-zero overlap integrals, on top of a sparse background of qps and qhs.

The final form of the structure factor up to second order in H_T is

$$S(Q, \omega) = \frac{2}{\pi} e^{-Q^2/4l_0^2} \left[(1 - M_Q^2) \delta(\omega_-(Q) - \omega) + M_Q^2 (\delta(\omega_+(Q) - \omega) + E(Q) \delta'(\omega_-(Q) - \omega)) \right] \quad (9)$$

Where $Q = 2l_0^2 q$, and $M_Q = |t_d|/4E(Q)$, and δ' denotes the derivative of the delta function with respect to its total argument. This latter term effectively doubles the matrix element of the secondary branch, such that the spectral weighting of the higher energy branch is $2M_Q^2$, and the lower branch has spectral weighting $1 - 2M_Q^2$. Solving for the excitation spectrum, we obtain two distinct branches given by

$$\omega_{\pm}(Q) = \Delta + V(Q) \pm E(Q) \quad (10)$$

Here, the splitting ϵ_d of the localized fermion has been neglected because in the relevant parameter regime it will be much smaller than the $E(Q)$ due to the relatively large distance between localized vortices. The splitting of the two excitation branches is a direct result of vortices supporting Majoranas, and the exponentially decaying splitting demonstrates that well separated Majoranas have approximately zero energy. These are two necessary ingredients for the vortices to be non-abelian quasiparticles. The expected dispersion is shown in Fig. 1. Experimentally, the most important signature is the crossing of the two roton branches near $ql_0 = 2$. As a consequence, the roton minimum has split into two and shifted to higher and lower q values, respectively. The parameters used for $E(Q)$ are outlined below Eq. (1). We have also taken the roton gap to be half the exciton gap $\Delta \approx 0.025e^2/\epsilon l_0$, the energy required to create a maximally separated qp-qh pair [19], which agrees well with exact diagonalization studies [20, 21], and have set the roton minimum to occur at $ql_0 \approx 1.4$, in good agreement with numerics [14].

The two quantities $E(Q)$ and t_d are independently tunable. The former determines the amount of splitting between the two branches and is a function of the separation of the dipole (which is proportional to the momentum at which the roton branch is probed). The latter, t_d , determines the spectral weighting of the secondary branch, but will also broaden both branches, and is a function

of the proximity of the dipole to the closest pinned vortex pair, and therefore is directly related to the disorder density.

The parameter t_d must be tuned such that $t_d \ll E(Q)$, so that the broadening of the excitation spectrum lines does not fuse the two branches. However, t_d must be sufficiently large that a measurably bright higher energy band, whose intensity is proportional to $|t_d|^2$ is observed.

In addition to the Moore-Read state, alternative wavefunctions have been proposed for the description of the $\nu = 5/2$ state. Most notably, the 331 candidate wavefunction differs from the Moore-Read state in that the spins are unpolarized. We can obtain information about the roton excitation of the 331-state by using the fact that this wavefunction has been used to describe pseudo-spin correlations in $\nu_{\text{tot}} = 1/2$ bilayer QH systems. For bilayer systems, the inter-layer Coulomb interaction takes the form $V(r) \propto (r^2 + d^2)^{1/2}$, where d is the interlayer spacing. The relevant case for $\nu = 5/2$ then, is to take the $d \rightarrow 0$ limit of these results. Two magnetoroton spectra have been predicted for $\nu_{\text{tot}} = 1/2$, due to in-phase and out-of-phase pseudo-spin density fluctuations [22], and these persist in the $d \rightarrow 0$ limit. If the 331 wavefunction correctly describes $\nu = 5/2$, then the following observations can be made. The splitting between the two branches is a minimum at the roton minima, contrary to our result. Depending on the splitting magnitude, the upper branch may be lost in the excitation continuum altogether. Furthermore, the splitting will not decrease exponentially (or even at all) with increasing q . Finally, the splitting will not oscillate. The combination of these signatures allows for a clear distinction from the splitting of the roton minimum for the Moore-Read state as described above.

Secondly, a ‘generalized composite fermion’ picture of $\nu = 5/2$ was recently discussed and illustrated by exact diagonalization [20]. In order to understand the lowest energy excitations, the possibility of two types of elementary excitations was assumed: a magnetoroton consisting of a qp-qh pair, and a broken composite fermion pair. This scenario for $\nu = 5/2$ was raised already some time ago [21]. From the numerical work [20], it seems possible that the gap for a broken pair type excitation may be slightly below the magnetoroton gap. However, the broken pair spectrum is a Bogolubov spectrum that monotonically increases with increasing q , while the magnetoroton spectrum on the other hand asymptotically approaches the exciton gap with increasing momentum. For momenta relevant to our current calculation then, that is for q beyond the roton minimum, the broken pair energy will be larger than the magnetoroton energy, and will disappear into the continuum region. Most importantly, composite fermions are charge neutral and are expected to couple only weakly to the charge density operator. For this reason, it is unclear whether the broken pair mode can be detected experimentally when studying the struc-

ture factor $S(q, \omega)$.

Finally, we comment on the neutral fermion gap: the energy required to add a single unpaired composite fermion to the system [23, 24], which is comparable to the exciton gap Δ [23]. The neutral fermion dispersion also contains roton-like minima [25], however an experimental probe must change the particle number parity [24]. Thus the collective excitation spectrum presented here will not include the neutral fermion dispersion.

We have shown that the magnetoroton dispersion of the $\nu = 5/2$ Moore-Read state splits into two distinct branches due to a fermionic degree associated with qp-qh pairs. This neutral fermion consists of a pair of Majorana states in the vortex cores which are responsible for the non-Abelian statistics of the Moore-Read state. In order to overcome the requirement of parity conservation, we coupled the qp-qh pairs to localized qps in the bulk of the quantum Hall state. The intensity of the secondary branch of the roton dispersion is proportional to the square of the coupling strength to these bulk qps. Experimental observation of this splitting by means of optical spectroscopy would constitute compelling evidence for the validity of the Moore-Read state and thus non-Abelian quasiparticles in the $\nu = 5/2$ QH state.

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